A NOTE ON THE EARLY WARNING SYSTEM OF CHANGE POINTS: COMBINATION OF REGIME SWITCHING AND THRESHOLD MODELS

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Abstract

Abrupt changes are a prevalent feature of financial data sets, such as prices of financial assets, returns of stocks, exchange rates, etc. An early warning system (EWS) can detect existing changes and predict possible future changes before they occur. Two important statistical models for change point detection and prediction are the regime-switching and threshold models. In the first model, the data set involves multiple structures that characterize the time series behaviours in different regimes. In a threshold model, change is detected as soon as a split variable passes a threshold. In this paper, by combining the two mentioned models, namely regime switching and threshold, an EWS for change point detection is designed. The underlying process for change detection obeys an AR(1) process. States are latent variables specifying whether a special time point is changed or not. They are realizations of the Markov chain. The predictive transition probabilities are determined by a threshold model based on adaptive recursive relations. This combination forms the mentioned EWS. Finally, two applications are given about change detection in stock returns and specifying business cycles.

Keywords: AR(1) process, abrupt changes, business cycles, Markov chain, regime-switching probabilities, split thresholds

JEL Classification: C53; C58; C63; G13; G17

1. Introduction

Many economic time series occasionally exhibit dramatic breaks in their behaviour, associated with events such as financial

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crises or abrupt changes in government policy. The time series of commodity prices, exchange rates, and macroeconomic indicators such as inflation and interest rates are always suspected that at some unknown points, their means or volatilities are changed rapidly or gradually. There are many statistical and econometric methods for modelling point-of-change problems. These methods include least squares, Bayesian, likelihood ratio, information criteria, non-parametric and parametric methods and quality control charts. The change point analysis in the univariate and multivariate time series, almost all types of econometric regression models, changes in the statistical distribution of variables such as mean, variance and covariance structure and even random graph models have been studied. Two main approaches for studying the change points are regime switching models and threshold analysis. For a comprehensive review of change point analysis, see Pons (2018) and references therein.

The Markov switching model of Hamilton (1994), also known as the regime-switching model, is one of the most popular nonlinear time series models considering the regime shift in economic models. Krolzig (1997) stated that a feature of the Markov switching model is that the switching equation is controlled by a latent state variable which obeys a first-order Markov chain. This model is perfect for describing dependent data showing dynamic patterns during different periods. Indeed, economists are interested in the behaviours of many economic variables, which are quite different during economic downturns.

The regime-switching model contains two components. First, x_t is observation at time t which satisfies in a time series model with an unknown parameter θ_{s_t}which depends on Markov chain s_t (a latent variable) with state space $\{0,1\}$ and transition probabilities

$$p_{ij} = P(s_{t+1} = j|s_t = i), i, j = 0,1.$$

Assuming, $[s_t = 1]$ stands for a bad event in the economy, such as a financial crisis or bankruptcy of important companies, then the posterior predictive probability

$$P(s_{t+h} = 1|x_t, x_{t-1}, \dots, x_1),$$

plays the role of the early warning system.

Quantities

$$P(s_t = 1|x_t, x_{t-1}, \dots, x_1),\ P(s_t = 1|x_T, x_{T-1}, \dots, x_1),$$

for some $T > t \geq 1$ are filtered and smoothed probabilities. For example, the event $[s_t = 1]$ says about the possible existence of an economic crisis at t −th year whereas x_t can be the level of inflation in a country, for example, represented by an RS-auto-regressive model. The x_t itself may be binary variable 0,1 related to hyperinflation or regular economic situations governed by a logit or probit regression. Hamilton filter (see Hamilton, 1994) proposes a recursive relation to make Bayesian inference about s_t given information $x_t, x_{t-1}, \dots, x_1$.

Similarly, some recursive relations are derived for smoothed and forecasting probabilities (see Kim, 1994). Basically, these recursive relations use the expectation maximisation (EM) algorithm to make inferences about s_t. These situations are studied by many authors in the early warning system literature, see Kole (2019). However, the novelty of this paper is that s_t is characterised by a threshold time series analysis.

The rest of the paper is organised as follows. In Section 2, literature reviews about the regime-switching model, threshold analysis techniques, as well as change point analysis are given. EWS models are also reviewed. Section 3 contains the methodologies applied in the current paper. Required notations and propositions are proposed. There first, the threshold states are defined. Then, the filtered, predictive and transition probabilities are given by using regime-switching estimates of the AR(1) coefficient as a state equation. Section 4 contains the data and data analysis in two data sets, that is, the change point detection in stock returns and the diagnosis of the business cycles. Section 4 concludes.

2. Literature review

The literature review covers three topics, including regime-switching models, early warning systems and threshold analysis.

The regime-switching models capture changes in the economic system that generates the data. Asako and Liu (2013) studied the potential ability of the regime-switching model to study the dynamic of inflation. They concluded that a regime-switching model with an independent shift in the mean and variance is best fitted to data and has minimum variance forecast with respect to other models. Pons (2018) specified that the Markov switching model also differs from the models of structural changes. The structural change models allow for frequent changes at random time points, but the latter admits only occasional and exogenous changes. Kapetanios (2003) stated that papers in the literature use time series techniques with two regimes and apply their models to their different economic time series, such as exchange rates. Krolzig (1997) showed that using the Markovian property, which regulates the current position of the state variable, a time series mechanism may prevail for a random period of time. The mentioned structure will be replaced by another structure when a switch takes place. Hamilton (1994) employed the Markov switching model to capture macroeconomic models of financial crises. They concluded that the Markov shifting model is found to successfully capture the timing of regime shifts in the financial/credit shocks. Recently, the regime-switching model has also been a popular choice in the study of business cycles. A highly successful attempt is incorporating a switching mechanism into the conditional variance of ARCH and GARCH models.

The regime-switching model, together with threshold analysis, is used to construct the early warning systems, which are referred to as early warning systems (EWS). There are many methods for making early warning systems. Sanchez-Espigares and Lopez-Moreno (2021) stated that the two well-known approaches for making the EWSs are signalling and logit and probit regressions methods. Gao *et al*. (2015) studied how EWS helps system managers to make correct and suitable decisions before faults occur.

Nonlinear time series such as threshold models have been studied by many papers in the literature for a comprehensive review of threshold analysis. An example is Asako and Liu (2013), who studied the effect of speculative bubbles on prices in the stock market of the United States, Japan and China using the threshold analysis at which the probability of breaking the bubbles related to the splitting variable. z_t and threshold ƙ. Throughout the TA model, the time series x_t has a structural break as soon as the splitting variable z_t goes beyond (or comes below) the threshold ƙ, see do-Dios Tena and Tremayne (2009), Kapetanios (2003) and references therein.

In the current paper, it is assumed that the least square estimate of the coefficient of regime shift first-order autoregressive AR(1) plays the role of splitting variable z_t which, as soon as it passes the specified threshold, then a change point occurs in the parameters of the time series x_t which is equivalent to the state $[s_t = 1]$. Otherwise, event $[s_t = 0]$ happens.

3. Methodologies

Here, using the threshold analysis technique, states of regime-switching models are defined. To this end, let x_t be the mean corrected first-order autoregressive AR(1) process x_t defined by

$$x_t = \beta_t x_{t-1} + \varepsilon_t; t \geq 1,$$

where x_0 is the initial value of the process, $|\beta_t| < 1$ for all t's,

$$\varepsilon_t = \sigma_t z_t,$$

at which z_t is white noise $WN(0,1)$ time series.

Indeed,

$$x_t = \beta_{s_t} x_{t-1} + \sigma_{s_t} z_t$$

Here, $\beta_t = \beta_{s_t}$ and $\sigma_t = \sigma_{s_t}$ represent the regime-switching process where s_t is a 0-1 valued Markov chain with transition probabilities

$$p_{ij} = P(s_t = j | s_{t-1} = i); i, j = 0,1.$$

Indeed, the regime-switched processes are

$$A_{s_t} = A_0 1(s_t = 0) + A_1 1(s_t = 1),$$

for $A = \beta, \sigma$ at which $1(s_t = i)$ is zero if $s_t = i$ and zero otherwise, for $i = 0,1$.

As previously mentioned, using the threshold analysis technique, the least square estimate of the coefficient β determines the state of the world.

Indeed, for example, if $\hat{\beta}_t > threshold$ (for some fixed but unknown constant $threshold$), then the time series data x_t comes from state $s_t = 1$, and as soon as $\hat{\beta}_t \leq threshold$, then $s_t = 0$ governs on process x_t. Since $\hat{\beta}_t$ depends on $\hat{\beta}_{t-1}$, then s_t is a first-order Markov chain with a transition matrix $P = (p_{ij})_{i,j=0,1}$. For example, $p_{10} = P\left(\hat{\beta}_t \leq threshold \middle| \hat{\beta}_{t-1} > threshold\right)$. Since $threshold$ is unknown, then s_t is a hidden Markov chain.

Similar to most regime-switching processes, the x_t is an observable process that its parameters depend on s_t. In the simplest form, it constitutes a first-order regime.

3.1. Regime switching coefficients

Suppose that the initial value of β_{s_t} is $\beta_0 = \beta$. Based on the first t observations, and if there is no change in initial value throughout t observations, the least square estimate of β is given by

$$\hat{\beta}_t = \frac{\sum_{i=2}^{t} x_i x_{i-1}}{\sum_{i=2}^{t} x_{i-1}^2},$$

The exponentially weighted least square estimate is given by

$$\hat{\beta}_{wt} = \frac{\sum_{i=2}^{t} \gamma^{t-i} x_i x_{i-1}}{\sum_{i=2}^{t} \gamma^{t-i} x_{i-1}^2},$$

for some suitable forgetting factor $\gamma \in (0,1)$. For practical applications, usually $\gamma = 0.95$ is chosen. This estimate gives higher weights to recent observations, which ensures that successive observations in the same state block are applied to calculate the time-varying (state-to-state) coefficients of the AR(1) process.

Define weights of recursive estimates of the coefficient of AR(1) by

$$\lambda_t = \frac{x_{t-1}^2}{\sum_{i=2}^{t} x_{i-1}^2},\ \lambda_{wt} = \frac{x_{t-1}^2}{\sum_{i=2}^{t} \gamma^{t-i} x_{i-1}^2},$$

and deviation of the least square estimate $\hat{\beta}_t$ (when there is no regime shift) of initial value β_0 denoted by $\hat{\theta}_{0t} = \hat{\beta}_t - \beta_0$.

Also, in the case of time-varying β's, let

$$\hat{\theta}_t^{tv} = = \hat{\beta}_t - \beta_t.$$

Under the regime-switching model, let $\hat{\theta}_t = \hat{\beta}_t - \beta_0$.

<u>*Proposition 1.*</u> Relations (*a*) - (*e*) are correct.

(*a*) $\hat{\beta}_t = (1 - \lambda_t)\hat{\beta}_{t-1} + \lambda_t \frac{x_t}{x_{t-1}}$,

(*b*) $\hat{\beta}_{wt} = (1 - \lambda_{wt})\hat{\beta}_{wt-1} + \lambda_{wt} \frac{x_t}{x_{t-1}}$,

(*c*) $\hat{\theta}_{0t} = (1 - \lambda_t)\hat{\theta}_{0t-1} + \lambda_t \frac{\varepsilon_t}{x_{t-1}}$,

(*d*) $\hat{\theta}_t = (1 - \lambda_t)\hat{\theta}_{t-1} + \lambda_t \frac{\varepsilon_t}{x_{t-1}} + \lambda_t s_t(\beta_1 - \beta_0)$,

(*e*) $\hat{\theta}_t^{tv} = (1 - \lambda_t)\hat{\theta}_{t-1}^{tv} + \lambda_t \frac{\varepsilon_t}{x_{t-1}} - (1 - \lambda_t)(\beta_t - \beta_{t-1}) =$

$$(1 - \lambda_t)\hat{\theta}_{t-1}^{tv} + \lambda_t \frac{\varepsilon_t}{x_{t-1}} - (1 - \lambda_t)(s_t - s_{t-1})(\beta_1 - \beta_0).$$

<u>*Proof.*</u> Parts (a)-(d) are easy to prove and they are omitted. For part (e), it is enough to see that

$$\beta_t - \beta_{t-1} = \beta_{s_t} - \beta_{s_{t-1}} = (s_t - s_{t-1})(\beta_1 - \beta_0)$$

3.2. Required probabilities

In this sub-section, transition, filtered and predictive probabilities are proposed.

(a) Transition probabilities.

Let ζ be a small positive number. Then,

$$p_{01}^{t} = P(\hat{\theta}_t > \zeta | \hat{\theta}_{t-1} = \zeta) = P\left(\frac{\varepsilon_t}{x_{t-1}} > \zeta - \delta\right),$$

where $\delta = \beta_1 - \beta_0$.

Therefore, one can see that

$$p_{00}^{t} = 1 - p_{01} = 1 - P\left(\frac{\varepsilon_t}{x_{t-1}} > \zeta - \delta\right)$$

$$p_{10}^{t} = P(\hat{\theta}_t = \zeta | \hat{\theta}_{t-1} > \zeta) = P\left(\frac{\varepsilon_t}{x_{t-1}} > \zeta\right),$$

and $p_{11}^{t} = 1 - p_{10}^{t}$. Here, $\varepsilon_t = \sigma_t z_t$. It is assumed that z_t has a standard normal distribution.

(b) Filtered probabilities.

Consider the filtered probability

$$\pi_t(1) = P(s_t = 1 | x_t, \dots, x_1),$$

which can be written as

$$\begin{aligned}\pi_t(1) = {} & P(s_t = 1 | s_{t-1} = 1, x_t, \dots, x_1) P(s_{t-1} = 1 | x_{t-1}, \dots, x_1) \\ & + P(s_t = 1 | s_{t-1} = 0, x_t, \dots, x_1) P(s_{t-1} = 0 | x_{t-1}, \dots, x_1) \\ & = p_{01}\pi_{t-1}(0) + p_{11}\pi_{t-1}(1)\end{aligned}$$

Also, $\pi_t(0)$ can be written $p_{00}\pi_{t-1}(0) + p_{10}\pi_{t-1}(1)$. Thus, assuming $\boldsymbol{\pi}_t = (\pi_t(0), \pi_t(1))^T$, it is seen that $\boldsymbol{\pi}_t = P_t^T \boldsymbol{\pi}_{t-1}$.

Notice that $\pi_t(1) = P(s_1 = 1 | x_1)$. Since, $x_1 = \beta_{s_1} x_0 + \varepsilon_1$, it is seen that

$$\frac{x_1}{x_0} - \beta_0 = \beta_{s_1} - \beta_0 + \frac{\varepsilon_1}{x_0} = \beta_1 - \beta_0 + \frac{\varepsilon_1}{x_0} = \delta + \frac{\varepsilon_1}{x_0}$$

Hence, replacing the current relation in the last above equation, one can see that

$$\pi_t(1) = P\left(\frac{\varepsilon_1}{x_0} > \zeta - \delta\right),$$

$$\pi_t(0) = 1 - \pi_t(1).$$

It is easy to see that

$$\boldsymbol{\pi}_t = \{\prod_{i=2}^{t} P_i\}^T \boldsymbol{\pi}_1$$

Assuming $\boldsymbol{\pi}_t$ converges to $\boldsymbol{\pi}_\infty$, as $t \to \infty$, then $\boldsymbol{\pi}_\infty = P_\infty \boldsymbol{\pi}_\infty$.

The distribution $\boldsymbol{\pi}_\infty$ is referred to stationary distribution and its existing, uniqueness, and finding it, is too important.

(c) Predictive probabilities.

The predictive probability

$$f_{t+1}(i) = P(s_{t+1} = i|x_t, \ldots, x_1), i = 0,1$$

are written as

$$f_{t+1}(0) = p_{00}\pi_t(0) + p_{10}\pi_t(1)$$

$$f_{t+1}(1) = p_{01}\pi_t(0) + p_{11}\pi_t(1)$$

Letting $\mathbf{f}_{t+1} = (f_{t+1}(0), f_{t+1}(1))^T$, it is seen that $\mathbf{f}_{t+1} = P_t^T\pi_t$. The h-step ahead prediction probabilities vector is given by

$$f_{t+h} = P_t^T \times \ldots \times P_{t+h-1}^T \times \pi_t$$

3.3. AMOC model

Here, the famous structural break model is studied using the above-mentioned results. Suppose that s_0 and configuration

$$(s_1, \ldots, s_{k_0}, s_{k_0+1}, \ldots, s_t) = (0,0, \ldots ,0,1,1, \ldots ,1)$$

is observed. Here, the lengths of zeros and ones are k_0, $t - k_0$, respectively.

This model is the at most one change (AMOC) model in change point analysis literature and k_0 is an unknown change point. The likelihood function is given by

$$L = p_{00}^1 \times \ldots \times p_{00}^{k_0} \times p_{01}^{k_0+1} \times p_{11}^{k_0+2} \times \ldots \times p_{11}^t$$

Suppose that $\sigma_{s_t} = \sigma$ is known and ε_t's are *iid* normal $N(0, \sigma^2)$ distribution. Then, p_{00}^t, p_{01}^t and p_{11}^t are given as in the following Table.

Table 1

Transition probabilities

Probability	Equation
$\mathbf{p_{00}^t}$	$\Phi\left(\frac{\zeta - \delta}{\sigma}\|x_{t-1}\|\right)$,
$\mathbf{p_{01}^t}$	$\Phi\left(\frac{\delta - \zeta}{\sigma}\|x_{t-1}\|\right)$,
$\mathbf{p_{11}^t}$	$\Phi\left(\frac{-\zeta}{\sigma}\|x_{t-1}\|\right)$.

Source: Author's

It is interesting to propose the maximum likelihood estimate of threshold parameter ζ. To this end, the likelihood function is given by

$$L = \prod_{i=1}^{k_0} \Phi\left(\frac{\zeta - \delta}{\sigma}|x_{i-1}|\right) \times \Phi\left(\frac{\delta - \zeta}{\sigma}|x_{k_0}|\right) \times \prod_{i=k_0+2}^{t} \Phi\left(\frac{-\zeta}{\sigma}|x_{i-1}|\right)$$

The following proposition summarizes the above discussion.

Proposition 2. Relations (*a*)-(*d*) are correct.

(*a*) The time-varying transition probabilities are given by p_{00}^t, p_{01}^t and p_{10}^t, as defined in Table 1.
(*b*) Let P_i be i-th probability transition matrix, then filtered probability $\boldsymbol{\pi}_t$ is $\boldsymbol{\pi}_t = \{\prod_{i=2}^{t} P_i\}^T \boldsymbol{\pi}_1$. This probability vector plays the role of EWS.
(*c*) Predictive probability is $\mathbf{f}_{t+1} = P_t^T \pi_t$,
(*d*) The likelihood function L, under the AMOC model, is

$$L = \prod_{i=1}^{k_0} \Phi\left(\frac{\zeta-\delta}{\sigma}|x_{i-1}|\right) \times \Phi\left(\frac{\delta-\zeta}{\sigma}|x_{k_0}|\right) \times \prod_{i=k_0+2}^{t} \Phi\left(\frac{-\zeta}{\sigma}|x_{i-1}|\right),$$

the MLE estimate of parameters such as ζ are obtained by numerical optimisation methods.

Table 2 below shows various values of the maximum likelihood estimate (MLE) of ζ, for $n = 525, k_0 = 273, \beta_0 = 0.25$.

Table 2

MLE of ζ

$\beta_1 \backslash \sigma$	**0.1**	**0.2**	**0.3**	**0.4**
0.35	0.203	0.101	0.075	0.041
0.45	0.221	0.151	0.111	0.205
0.55	0.251	0.188	0.158	0.215
0.65	0.296	0.197	0.182	0.255
0.75	0.304	0.204	0.195	0.286
0.85	0.321	0.213	0.201	0.304

Source: Author's

4. Data analysis

This paper combines regime-switching models with threshold analysis to obtain better results in change point analysis, a type of early warning system frequently used in financial fields. This manuscript proposes useful propositions to find heading warning probabilities. Here, two applications of the above-mentioned theoretical results are given in stock returns and business cycles.

(a) Stock returns.

Returns of any stock may be positive or negative, which are states of the fitted Markov chain in this sub-section. Local trends are successive short-length sequences of positive or negative returns. Finding these local trends, in the short run, is too important for traders and scalpers to take long or short positions. Here, a regime-switching model with fixed transition probabilities is fitted to the daily return of *Amazon Co*.

This series is defined as follows

$$x_t = \log(\mathrm{S}_t) - \log(\mathrm{S}_{t-1}),$$

for a period of 25 May 2021 to 23 May 2023, including 505 observations. Here, S_t denotes the price of the stock of *Amazon Co*. at $t\text{-}th$ day. The one-step transition probabilities matrix is

$$\begin{bmatrix} 0.53 & 0.47 \\ 0.45 & 0.55 \end{bmatrix}$$

The slopes of AR(1) for negative and positive returns are -0.00164, and 0.03489, respectively. The volatility seems to be fixed by examining the rolling estimates of standard deviation and checking its stability during the time, and it is 0.026. Here, using the Monte Carlo simulation, the distribution of maximum M of $x_t, t = 1, \dots, 505$ is approximated. It is seen that. $M^{0.5}$ has zero skew and three kurtosis. That is, the normal distribution with a mean of 0.284 and standard deviation of 0.0153 is proposed for this random variable. Let τ be the upper bound for M with a 0.95 confidence level. Then, $\tau = 0.3091$.

(b) Business cycles.

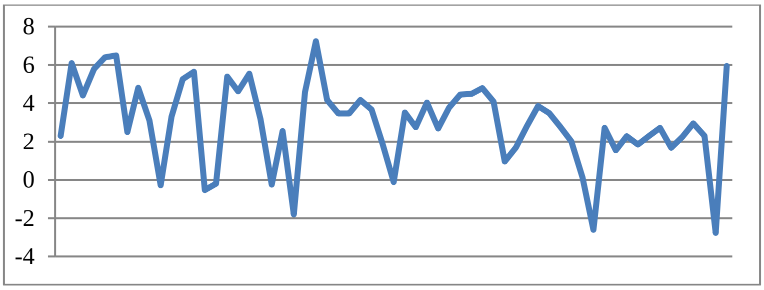
In economics, the business cycles are distinguished by the growth rate of macroeconomic variables such as unemployment and GDP. To this end, compute

$$x_t = \log(\mathrm{GDP}_t) - \log(\mathrm{GDP}_{t-1}),$$

where GDP_t is the GDP of $t\text{-}th$ year. The data set is the growth rate of the US GDP from 1960 to 2020 (62 observations), taken from https://data.worldbank.org/. A time series plot indicating three cycles is proposed.

Figure 1

Time series plot of GDP-GR of US

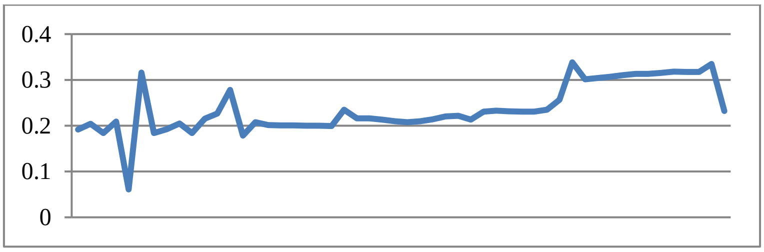


Source: Author's contribution

The first-order AR is fitted, and the time series plot of $\hat{\beta}_t$ is given as follows, which clearly indicates a change.

Figure 2

Slope of AR(1) model

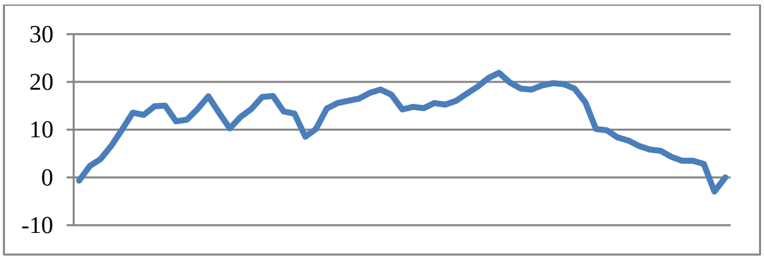


Source: Author's contribution

The logarithm of the likelihood function indicates the inverse V-shaped figure. With a high probability, there is a possible change point at $t = 40$ where the log-likelihood has been received to its maximum.

Figure 3

Log-likelihood plot

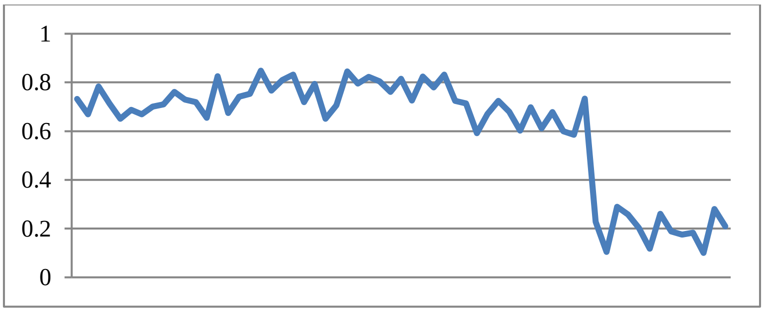


Source: Author's contribution

To make sure about the existence of the change point and its location, the probabilities p_{00}^t are plotted as follows. From this plot, it is seen that after t = 40, the probability of moving to state zero from state zero is negligible. Therefore, it is suspected there exists a change point at $t = 40$.

Figure 4

Plot of p_{00}^t



Source: Author's contribution

5. Concluding remarks

This paper studies the relationship between regime switching and threshold models and designs EWS using this relationship. The underlying process is AR(1). Some theoretical perspectives of EWS are proposed. Finally, it uses this EWS to study trading patterns and identify turning points in the stock market. This article has the following differences and advantages compared to similar articles.

a) Although Asako and Liu's (2013) paper deals with the identification of price bubbles from the point of view of change, it does not look at the issue from the perspective of regime change, a structure that mainly occurs in the stock market.
b) The paper by Gao, Cecati, and Ding (2015) is mainly used to identify change and fault points in mechanical systems. Although these methods are conventional, since they mainly consider deterministic methods, they cannot analyse the system's random states.
c) Do-Dios Tena and Tremayne (2009) did not use threshold models in the analysis, which are mostly necessary in stock data analysis.

d) Although Hamilton's book (1994) is very good at describing all the variables of financial time series, it does not provide combined methods.
e) Kapetanios (2003) considers only regime-switching models for change point analysis, which requires early warning algorithms, which are not found in that paper.

References

1. Asako, K., and Liu, Z. (2013). A statistical model of speculative bubbles with applications of the stock market of the United States, Japan and China. *Journal of Banking and Finance*, 37(7), pp. 2639-2651. https://doi.org/10.1016/j.jbankfin.2013.02.015.
2. do-Dios, Tena, J., and Tremayne, A.R. (2009). Modelling monetary transmission in the UK manufacturing industry. *Economic Modelling* 26(5), 1053-1066. https://doi.org/10.1016/j.econmod.2009.04.003
3. Gao, Z., Cecati, C., and Ding, S.X. (2015). A survey of fault diagnosis and fault-tolerant techniques Part I: Fault diagnosis with model-based and signal-based approaches. *IEEE Trans. Ind. Electron*. 62(6), pp. 3757-3767. doi: 10.1109/TIE.2015.2417501
4. Hamilton, J.D. (1994). *Time Series Analysis.* Princeton University Press.
5. Kapetanios, G. (2003). Threshold models for trended time series. *Empirical Economics, 28*, pp. 687-707.
6. Kole, E. (2019). Markov switching models: an example for a stock market index. *Working Paper Econometric Institute, Erasmus School of Economics*, Erasmus University Rotterdam. Available at: https://dx.doi.org/10.2139/ssrn.3398954
7. Krolzig, H.M. (1997). Markov-switching Vector Auto-Regressions: Modelling, Statistical Inference, and Application to Business Cycle Analysis. Berlin: Springer.
8. Kim, C.J. (1994). Dynamic linear models with Markov-switching. *Journal of Econometrics*, 60 (1-2). pp. 1-22. Available at: https://doi.org/10.1016/0304-4076(94)90036-1.
9. Pons, O. (2018). *Estimations and Tests in Change-Point Models.* World Scientific Press. https://doi.org/10.1142/10757
10. Sanchez-Espigares, J.A., and Lopez-Moreno, A. (2021). MSwM examples. *Technical Report*. *Department of Statistics and Operations Research. UPC-Barcelona Tech*. Available at: https://cran.r-project.org/web/packages/MSwM/