A COPULA-GARCH MODEL FOR A PROXY PORTFOLIO FOR BET-FI INDEX

Marius ACATRINEI, PhD *

Abstract

The paper fits a copula-Garch model for a proxy portfolio of BET-FI index and computes its Expected Shorfall. We used daily closing prices spanning for a two year period. The results indicate that the portfolio's Expected Shortfall computed with a copula-Garch is higher than otherwise reported by a naive Value at Risk. The VaR computed with variance-covariance method assumes a multivariate Gaussian distribution and produces results that constantly underestimates the risk due to incorrect distributional assumptions.

Keywords: Garch model, Copula function, Stock Market Indices, Expected Shortfall

JEL Classification: C32, C51

1. Introduction

In the financial econometrics literature it is accepted the idea that the equity returns are not normal, and are not independent. This stylized facts have consequences on the calculation of other risk measures for portfolio returns. For example the Value at Risk of a portfolio is usually computed with the variance-covariance matrix, but such a method is suitable only for elliptically distributed random variables. Since the equity returns are not elliptically distributed, the calculated risk measure (Value-at-Risk/Expected Shortfall) will underestimate the risk of the portfolio. Also by assuming a Gaussian multivariate distribution the tail risk is underreported.

The Garch models carried out good results in the financial literature when the task is the modelling of financial returns which usually have extreme returns.

In computing Value at Risk estimates, the Basel II Accord suggested a risk assessment for the next ten days, which is usually done by a scaling factor named "square root of time"; in practice what is meant is that the standard deviation of a security (portfolio of securities) is scaled up by a factor of $\sqrt{10}$. Using a Garch model we may draw 10 day forecast of volatility which are desirable to use since

* *Economist, Financial Supervisory Authority, Romania.*

the scale factor underestimates volatility unless the volatility is a homoscedastic process.

Moreover when using the variance-covariance approach to an asset portfolio we assume that the marginal distribution are the same, while in practice it may be easily seen that they are not. By using a copula-Garch model it is possible to use different distributions for the margins. Therefore a copula function may deliver better results since in this way you can distinguish between the dependence structure and the marginal distributions. The copula is a measure of dependence between two or more variable.

The copula-Garch model was introduced by Rockinger and Jondeau (2002). Since only the marginal distributions are known for the equity returns, the authors used a copula function for linking the univariate models. Therefore the copula–Garch model allows the marginal distribution to be conditionally dependent through their dependency parameter, which is akin to the correlation parameter.

Patton (2006) investigated whether the exchange rates have a symmetric dependence structure which is assumed when using a bivariate normal or Student distribution. Dependency structures are created when the equity indices are falling due the joint increase in the risk aversion on the same/different market/s. Jondeau and Rockinger (2006) showed that dependency between European markets increased especially when the markets traded in the same direction. The authors captured the dependency structure using a Markov switching model. Micocci and Masala (2004) applied copula methods on the estimation of Value at Risk. Their research was applied to Italian equities companies. Embrechts and Dias (2003) showed the usefulness of Clayton's copula and survival copula while applied to the dependence structure of the tail of bivariate financial return in a study on high-frequency exchange rate returns. Hsu et al. (2008) proposed the use of the copula – Garch model for the estimation of the optimal hedge ratio as opposed to the regression based static approach. Chiou and Tsay (2008) used a copula model to price exotic derivatives and to assess Value at Risk of different assets.

2. A copula-Garch model

We use a copula-Garch model uses the marginal distributions of the returns and a Student t copula function to link the univariate models. The estimation of a copula-Garch model is carried out in two steps. In the first step a Garch model is fitted on the financial returns and the i.i.d standardized residuals obtained from filtration are transformed in uniform U(0,1) variables. In the second step the copula parameters are estimated.

The Garch model

A simple Garch model (1,1) may be written as

$$(1)\ r_t = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$$

$$(2)\ \mu_t = E(r_t | F_{t-1})$$

$$(3)\ \sigma_t^2 = Var(r_t | F_{t-1})$$

$$(4)\ \sigma_t^2 = E[(r_t - \mu_t)^2 | F_{t-1}]$$

$$(5)\ \sigma_t^2 = Var(\varepsilon_t | F_{t-1})$$

$$(6)\ \varepsilon_t = \sigma_t z_t \text{ unde } z_t \text{ sunt i.i.d. cu } N(0,1)$$

$$(7)\ \sigma_t^2 = \omega + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

At this point we have to make assumptions about the distributions of the errors. The Gaussian distribution, the Student's t, the skewed Student-t or GED distribution are usually used in practice. We have fitted a Garch(1,1) model with Student-t innovations.

By applying MLE or QMLE estimation procedure, we may estimate the unknown model parameters for each , k' process

$$(8)\ \zeta_k = \mu_k, \alpha_k, \beta_k, \omega_k, etc$$

The number of parameters model may vary depending on the Garch model employed and the distribution assumed.

The copula function

If the returns do not have elliptical distributions, then the Pearson's linear correlation is a misleading measure of association between the returns series. Rank correlations are non-parametric measures between the data based on ranked data. Spearman's rho and Kendall's tau are two rank correlation measures usually used in financial econometrics. Rank correlations are less sensitive to outliers.

For a sample with *n* observations we compute Kendall's tau by comparing all possible pairs of observations.

$$(9)\ \tau = \frac{N_C - N_D}{\frac{n(n-1)}{2}}$$

where N_C are the concordant pairs and N_D are the discordant pairs. A pair of two variables is said to be concordant if x_1-x_2 has the same sign as y_1-y_2 for the continuous random variables X and Y.

Since the estimate of the Kendall's tau provides only the sample estimate of the return series, in order to define the population parameters a copula function is required for the joint distribution of the return series.

Using a copula function, the joint distribution of 'k' processes may be written as

$$(10)\ F(r_1, \ldots r_k; \xi) = C\left(F_1(r_1; \zeta_1), \ldots F_k(r_k; \zeta_k)\right)$$

where ξ is the parameter vector of the copula function and $F_1(\cdot), \ldots F_k(\cdot)$ are the marginal distributions. The joint density is expressed as the product of the marginal distributions $f_1(\cdot), \ldots f_k(\cdot)$

A Student t copula is defined as

$$(11)\ C_v(u_1, \ldots, u_n; \Sigma) = t_{\Sigma,v}(t_v^{-1}(u_1), \ldots t_v^{-1}(u_n))$$

where $t_{\Sigma,v}$ is a multivariate Student t distribution, t_v is an univariate Student t distribution with v degrees of freedom and Σ is the correlation matrix.

Estimation of the copula-Garch model

In the first step we estimated the parameters of the Garch model and determined the standardized residuals. At this step there is a set of tests to determine if the chosen Garch model is adequately capturing the underlying process. At this step a different model for modelling the financial returns may be selected.

In the second step we used the standardized residuals and calculated pseudo-uniform variables. This computing step may be carried out either parametrically from the assumed distribution or semi-parametrically from the empirical distribution function. Using the pseudo-uniform variable we estimate a copula model. Using the dependence structure determined by the estimated copula we generated a number of (N) data sets of random variates for the pseudo-variable uniformly distributed. After the N date sets were generated, we have computed the quantiles (95%) for the Monte Carlo draws. Since we have a weight vector for our portfolio, we use the quantiles in order to calculate the N portfolio return scenarios. In the final step we used the N portfolio scenarios to calculate the Expected Shortfall for a given confidence level usually 95% or 99%.

$$(12)\ ES_\alpha = E[r_t | r_t \leq VaR_\alpha]$$

3. Estimation results

Our dataset includes BET-FI index and Fondul Proprietatea (symbol:FP), SIF2, SIF3, SIF4. The closing prices for BET-FI index and Fondul Proprietatea (symbol:FP), SIF2, SIF3, SIF4 were extracted from Bucharest Stock Exchange website (www.bvb.ro) from January 2013 to April 2015. Daily returns are calculated from the closing prices according to the formula $R_t = \ln(P_t / P_{t-1})$ where P_t is the daily closing price of the index.

From the structure of the BET-FI index we have constructed a portfolio consisting of Fondul Proprietatea (symbol:FP), SIF2, SIF3, SIF4 between January 2013 and April 2015. We have constructed four different portfolios by varying the weights of the companies. In this way we achieve two results: 1) we carried a sensitivity analysis and 2) we compared our results with a standard VaR.

The log returns of the indices showed excess kurtosis and negative skewness meaning that the data are heavy-tailed and not normally distributed. The characteristics of the equity returns series justified the investigation with a GARCH model.

Since the GARCH models work only with stationary time series, we have tested the log-returns of the indices with the Augmented Dickey Fuller (ADF) test and the Kwiatkowski–Phillips–Schmidt–Shin (KPSS) test in order to estimate the presence of a unit root.

The ADF tests rejected the null hypothesis of a unit root process and the KPSS test accepted the null hypothesis of a stationary process. Both tests come to the same conclusion for all time series at high significance levels. We did not present tables with the test statistics, the p-values and the used lag orders for both stationarity tests for presentation reasons.

We have estimated the returns using a simple GARCH (1,1) model with Student distribution for the errors. For each model all coefficients are significantly different from zero and the stability requirement for GARCH(1, 1) models is not violated, that is, $\alpha + \beta < 1$. The estimates for the degrees of freedom parameter (v) have pointed out a distinct deviation from normality. Therefore all the estimates have heavy tails distribution. The constant in the mean (μ) and in the variance (ω) equation are not significant for the all the series except for the constant in the variance equation for SIF2. The Garch(1,1) fitted results are reported in Table 2 in the Annex.

We have extracted the standardized residuals and performed one-step-ahead forecast on the conditional variance in order to use them in the calculation of the pseudo-uniform variable.

Since a Gaussian copula does not allow for dependencies in the tails, we have estimated a Student-t copula based on Kendall'rank correlation. Then we have generated 100.000 random sets and we determined the spectrum of simulated losses for each company.

We have repeated the volatility forecasts for each simulation of random sets and then we have simulated the portfolio losses. The results were sorted by size and the expected shortfall at the 95% confidence level was expressed as the median of the 50.000 largest losses.

We have constructed four portfolios by varying the weights of the four companies and we calculated the Expected Shortfall for each of them using a copula-Garch model. Finally we have compared the results with a naive Value at Risk computed with variance-covariance method. The results are reported in table 1.

Table 1

Expected Shortfall

	FP	SIF2	SIF3	SIF4	Expected Shortfall (95%)	Value at Risk (95%)
	Weights					
1.	0.25	0.25	0.25	0.25	1.68%	1.57%
2.	0.35	0.35	0.15	0.15	1.54%	1.46%
3.	0.15	0.15	0.35	0.35	1.88%	1.74%
4.	0.05	0.25	0.35	0.35	2%	1.83%

By comparing the variance-covariance VaR results with the copula-Garch results, it results that when using a model which assumes that the equity returns are i.i.d and Gaussian distributed, the reported risk measure will underestimate the risk of the portfolio.

4. Conclusion

In this paper we have applied a copula-Garch model for computing the Expected Shortfall, a risk measure, for a proxy portfolio of the BET-FI equity index. Usually the portfolio risk measures such as Value-at-Risk or Expected Shortfall are computed using restricting distributional assumptions. Using a copula-Garch model some of these assumptions are relaxed and the results should show to which degree are they underreporting the risk of the portfolio.

Instead of assuming a multivariate distribution, a copula function may by used instead for modelling the dependence structure and the marginal distributions between returns. Since the equity returns are heavy tailed we have fitted a GARCH (1,1) model with a Student-t distribution.

The copula function copula is a measure of nonlinear dependence between two or more variable. By distinguishing between the dependence structure and the marginal distributions, a copula function will capture better the tail risk between the financial returns. When calculating risk measures such as Value at Risk or Expected Shortfall we are primarily interested in the estimation of the tail risks.

The results suggest that a copula-Garch model is to be preferred in practice to the variance-covariance Value at Risk measure. Although the differences are small, it should be noted that the evolution of the Romanian capital market for the last two years could be labelled as being in a normal market condition, meaning that it was characterized by a low volatility regime. The gap between the two methods should be markedly different under distressed market conditions. Also we should note that the VaR calculated with the variance method will lead to wrong results by constantly underestimating the risk due to wrong distributional assumptions.

Further work will be devoted for computing Expected Shortfall/Value at Risk with a time-varying copula function which allow for the parameters of the function to vary over the sample period.

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ANNEX

Table 2

Fitted Garch(1,1) models

Company	Estimate	Std.Error	T value	p value
FP	-0.0001	0.0002	-0.4240	0.6717
μ	0.0000	0.0000	2.0170	0.0437
ω	0.0001	0.4543	1.9610	0.0499
α	0.0167	0.1167	2.7150	0.0066
β	0.891	0.3643	7.0590	0.0000
υ	2.2573	0.0002	-0.4240	0.0000
SIF2				
μ	-0.0004	0.0004	-0.8640	0.3874
ω	0.0000	0.0000	1.8950	0.0581
α	0.1773	0.0759	2.3360	0.0195
β	0.6750	0.1262	5.3500	0.0000
υ	3.8550	0.6593	5.8470	0.0000
SIF3				
μ	-0.0006	0.0005	-1.2940	0.1957
ω	0.0000	0.0000	2.1860	0.0288
α	0.1734	0.0844	2.0540	0.0400
β	0.6965	0.1105	6.3040	0.0000
υ	3.5810	0.6052	5.9170	0.0000
SIF4				
μ	0.0000	0.0004	0.0990	0.9208
ω	0.0000	0.0000	1.8180	0.0691
α	0.1456	0.0612	2.3800	0.0173
β	0.7547	0.0899	8.3960	0.0000
υ	3.7190	0.6503	5.7190	0.0000